



Feeling Safe and Secure in a Learning-Enabled Cyber-physical World

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About myself...

- Postdoctoral Scholar in ECE/CS at UCLA
 - NESL Research Group
- PhD in Computer Engineering with Cybersecurity Track at Rutgers
 - 4N6 Research Group
 - GAANN Fellow (Graduate Assistance in Areas of National Need)
- BS and MS in ECE from the University of Miami
- Academic Intern at CMU with André Platzer
- Previous Intern at Siemens Corporate Research









What does it mean to feel "safe" and "secure"?

Cyber-Physical Industrial Control Systems (ICS)



Water Treatment



Factory Automation



Electric Power Grid



Nuclear Reactor

Industrial Control System (ICS) Attacks



Programmable Logic Controllers (PLCs) and Industrial Control Systems (ICS)



Stuxnet/BlackEnergy Attack Overview

	BlackEnergy3 plugins	functionality	
	fs.dll	File system operations	
	jn.dll	Parasitic infector with a given payload	
	ps.dll	Password stealer	
	si.dll	System information	
	ss.dll	screenshots	DIC
	vs.dll	Network discovery and remote execution	FLC
	up.dll	Update malware	
	tv.dll	TeamViewer	STL code block
	dc.dll	List Windows accounts	
	bs.dll	Query system HW, BIOS, Windows info.	
	dstr.dll	Destroy system	
	kl.dll	Key-logger	
	scan.dll	Network host port scan	
	rd.dll	Simple pseudo 'remote desktop'	L- M II 12 I3 I4 I6 I6 I7 I8 12/24V INPUT 8xDC 17 = AI1 (0.10V) I8 = AI2 (0.10V)
	grc.dll	Back comm. channel using plus.google.com	
	cert.dll	certificate stealer	
	sn.dll	Logs traffic, extracts login-passwords	
	usb.dll	gathers info. on connected USB devices	

Observation: BlackEnergy3 – compared to Stuxnet - included many more reconnaissance plugins!

Trusted Safety Verifier (TSV) Overview



Warning! Violation Point in the Source Code

Infrastructural Safety Requirements

- Formulated using linear temporal logic (LTL) expressions
- Example safety requirement
 - English expression
 - Relay R₁ should **NOT** open **UNTIL** Generator G₂ turns on
 - Logical expression
 - Atomic propositions
 - a₁: "Relay R₁ is open"
 - a₂: "Generator G₂ is on"



Limitations of Previous Solutions

- Application layer protection
 - No means of protecting lower levels of abstraction
- Physical safety properties are assumed to be provided





Hey, My Malware Knows Physics! Attacking PLCs with Physical Model Aware Rootkit

NDSS 2017









Harvey: Model-Aware Rootkit (NDSS 2017)

- A rootkit that takes into account the physical topology of the ICS
- Model
 - Uses physical models to optimize control commands for an adversarial objective function
- PLC infection: compromising the PLC's firmware
 - Utilize the firmware update mechanism to replace firmware over the network
 - Local firmware modifications, e.g., SD card or JTAG implantation
 - Run-time attacks, e.g., network exploits or remote code execution vulnerabilities (FrostyURL)



Does not require 1. 1. **Compromised PLC** Stuxnet vs. Harvey **SCADA compromise to** control logic remain stealthy 2. **Modified HMI to prevent** Can calculate fake data 2. detection of PLC for dynamic systems, modifications PLC **Stuxnet only replays Actuator/Sensor** 3. Replayed benign recorded measurements to HMI measurements cyber **Supervisory Control (SCADA)** physical I/O

Stuxnet:

PLC: Programmable Logic Controller HMI: Human Machine Interface Harvey:



Stuxnet vs. HelmaspisrBthsedWide (Stoketielled), NDSS '14)

PLC Architecture & Adversary Model



Pby streature ss: Ad Weas a Data dollahipulation



Physics-Awareness: 2-Way Data Manipulation



CompactLogix L1 PLC

- High Value (1) ~ 24 V DC
- Low Value (0) ~ 8 V DC



Memory Analysis with JTAG



Memory Analysis with JTAG

- JTAG interface to dump memory for code disassembly
- TI Stellaris LM3S2793 data sheet to find memory layout and built-in ROM functions



Modified GPIO-Output Update ISR



Harvey Spoofing Inputs



Manipulation of Sensor Data

Harvey Spoofing Outputs



Real-world Attack Demo: Attacking a Power System





Real-world Attack Demo: Attacking a Power System

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Let's Get Defensive

 Problem: Legacy devices (such as PLCs) don't support a hardware trusted computing base (TCB) for remote attestation



Patt: Physics-based Attestation of Control Systems



Patt: Physics-based Attestation of Control Systems

(RAID 2019)











Untrusted



Untrusted

Let's Get Defensive

- Problem: Legacy devices (such as PLCs) don't support a hardware trusted computing base (TCB) for remote attestation
- **Problem:** The complexity of these CPS involves several domain-specific physical processes that need to be verified





See No Evil, Hear No Evil, Feel No Evil... *Print* No Evil?

Malicious Fill Pattern Detection in Additive Manufacturing

USENIX Security 2017





Industrial 3D Printing







Malicious Fill Pattern Detection (USENIX Security 2017)

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Let's Get Defensive

- Problem: Legacy devices (such as PLCs) don't support a hardware trusted computing base (TCB) for remote attestation
- **Problem:** The complexity of these CPS involves several domain-specific physical processes that need to be verified
- **Problem:** How can we verify these cyberphysical properties in distributed settings?





Control Behavior Integrity for Distributed Cyber-Physical Systems

ICCPS 2020









SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN



Scadman: Control Behavior Intrusion Detection

- An intrusion detection solution for distributed ICS
- Hybrid model
 - Uses physical state estimation for IDS
 - Updates physical state
 estimation based on software
 control flow



Scadman Overview





Evaluation: Water Treatment Testbed

- Evaluated against known set of ICS attacks from
 - 7 days worth of data
 - Multi-point attacks included
- Detected all attacks
 - Also detected faulty sensor data
 - Zero false positives
- No overhead on ICS operation
 - Scadman utilizes historian data





HyPLC: Hybrid PLC Program Translation for Verification

ICCPS 2019 (Best Paper Finalist)







Hybrid Systems Modeling of CPS

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Moving Away from Linear Temporal Logic (LTL)

Consider previous example

• LTL expression: Globally(L<TankLevel<H)



- Infinite amount of states for real-time systems
 - Can only provide runtime checks
- Real systems have non-deterministic states

How can we guarantee safety forever for complex CPS?

Safety: L<TankLevel<H

Verification Using Differential Dynamic Logic

 $[\alpha]\psi$ For all states in α , ψ holds true (safety)

 $\langle \alpha \rangle \psi$ There exists a state α where ψ is true (liveness)

Definition (Hybrid program α)			
x'=f(x)	(continuous evolution)	
$x := \theta$	(discrete jump)		
? χ	(conditional execution)		
lpha; eta	(seq. composition)		
$\alpha\cup\beta$	(nondet. choice)		
$lpha^*$	(nondet. repetition)		

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Verification Using Differential Dynamic Logic



HyPLC: Hybrid Programmable Logic Controller Program





Revisiting the PLC Scan Cycle



Revisiting the PLC Scan Cycle



* Original model would have reported a violation for this entire sequence based on the water tank level



* HyPLC model reports this same sequence as safe operation based on flow rate





Formal Guarantees for Macroprogramming Heterogeneous IoT Networks



Smart City Environment

Officer to IoT Network: "Identify shooter and follow"

*How do we expose device services safely?

DDFlow: Visualized Declarative Programming for Heterogenous IoT Networks (IoTDI 2019)



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Formal Guarantees for Macroprogramming Heterogeneous IoT Networks

Types of Policies				
P1: Mutually exclusive states mus	st not exist in the environment.	P2: User-defined rules.		
Racing Events	Cyclic Events	E.g.: Doors and windows must be locked is user is not home		
E1: CO ₂ density-high -> turn-on-fan	E1: user-home, lights-on -> lights-off	E1: user-away -> user-away-mode-on		
E2: temp-low -> turn-off-fan	E2: user-home, lights-off -> lights-on	E2: user-away-mode-on, temp-high -> windows-on		
CO ₂ density-high turn-on-fan	User-home lights-on lights-off	P: <user-not-home, doors-lock,="" windows-lock=""> temp-high open-windows</user-not-home,>		

Formal Guarantees for Macroprogramming Heterogeneous IoT Networks



Characterizing Security in IoT Macroprogramming Environments



The Case for Robust Adaptation: Autonomic Resource Management is a Vulnerability (MILCOM 2019)

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Let's Talk Through Physics! Covert Cyber-Physical Data Exfiltration on Air-Gapped Edge Devices (Submitting to USENIX 2020)

Robust Multi-modal Inferencing in Heterogenous IoT Environments



DeepCEP: Deep Complex Event Processing Using Distributed Multimodal Information (SMARTCOMP 2019)

Robust Multi-modal Inferencing in Heterogenous IoT Environments



Example: Detecting an unattended bag



Some future applications...



3D Trajectory and Orientation Estimation of Marine Animals



Some future applications...



Decoding How Humans Encode Memory

Seme

UCLA







Conclusion

- Practical security analysis based on physical properties of CPS
 - Harvey: a physics-aware, two-faced rootkit malware
 - Physics-aware defenses for at various levels of distribution as well as in domain-specific scenarios
- Applicability of **formal deductive verification** techniques in complex CPS
 - HyPLC: a bi-directional translation of PLC controller code and differential dynamic logic hybrid programs
- Practical security and safety considerations for learning-enabled IoT/CPS
 - Formal guarantees for macroprogramming heterogenous IoT networks
 - Multi-modal inferencing in distributed and heterogenous IoT networks

• Future Work

- Scalable deductive security verification via compositional verification and safety contracts
- Robust multi-modal inferencing in distributed and contested environments
- Security detection and intervention in IoT macroprogramming environments





Publications

- Data Flow Security for Mobile Devices
 - Context Aware Information-Flow-Based Micro-Security Perimeters for Mobile Devices. DSN, 2016
- Cyber-physical Vulnerability Assessment
 - A Cyber-Physical Modeling and Assessment Framework for Power Grid Infrastructures. *IEEE Transactions on Smart Grid, 2015*
 - Covert Channel Communication Through Physical Interdependencies in Cyber-Physical Infrastructures. IEEE SmartGridComm, 2014
 - Threat Model Quantification in Smart Grid Critical Infrastructures. IEEE SmartGridComm 2014
- Embedded Systems Verification and Controller Security
 - Hey, My Malware Knows Physics! Attacking PLCs with Physical Model Aware Rootkit. NDSS, 2017
 - Detecting PLC Control Corruption via On-Device Runtime Verification. IEEE Resilience Week 2016
- Cyber-physical Control Flow Integrity
 - Cyber-Physical Control Flow Integrity for Distributed Controllers. Submitting to ICCPS 2019
 - PAtt: Physics-based Attestation of Control Systems. RAID 2019
 - Tell Me More than Assembly Instructions! Reversing Semantics of IoT Software Binaries. DSN 2019
 - See No Evil, Hear No Evil, Feel No Evil, Print No Evil? Malicious Fill Patterns Detection in Additive Manufacturing. USENIX Security, 2017
- Hybrid Systems Modeling and Verification
 - HyPLC: Hybrid Programmable Logic Controller Program Translation for Verification. ICCPS 2019 (Best Paper Finalist)
- Macroprogramming of Distributed and Heterogeneous IoT/CPS
 - Let's Talk Through Physics! Covert Cyber-Physical Data Exfiltration on Air-Gapped Edge Devices. Submitting to USENIX Security, 2020
 - DDFlow: Visualized Declarative Programming for Heterogeneous IoT Networks. *IoTDI 2019*
 - RemedIoT: Remedial Actions for Internet-of-Things Conflicts. BuildSys 2019
- Robust Multi-modal Inferencing
 - PhysioGAN: Training High Fidelity Generative Model for Physiological Sensor Readings. *Submitted to TPAMI 2019*
 - DeepCEP: Deep Complex Event Processing Using Distributed Multimodal Information. SmartComp 2019
 - RadHAR: Human Activity Recognition from Point Clouds Generated through a Millimeter-wave Radar. MMNets 2019

Formal Verification of Hybrid Controller Logic for Transient Stability

(Case Study)





Transient Stability of a Simple Power System

Final Complete SMIB Hybrid Program *init* \Rightarrow [{*ctrl*; *plant*&*H*}*](*req*) *init* $\equiv P_M = 1 \wedge P_{e,max} = \frac{3}{2} \wedge \omega = 0 \wedge \theta = \arcsin(\frac{P_M}{P_{e,max}})$ $\bigwedge \theta_{max} = \pi - \theta \bigwedge \sin \theta = \frac{P_M}{P_{e_{max}}} \bigwedge \cos \theta = \frac{\sqrt{P_{e_{max}}^2 - P_M}}{P_{e_{max}}}$ $\Lambda c = 2P_M \theta_{max} - 2P_{emax} \cos(\theta)$ $ctrl \equiv \{(a := P_M - P_e, max\sin(\theta)) \mid | (a := P_M)\};$ t := 0;d0 := d; v0 := v;T := *: $?T \geq 0 \land \theta_f(T) \leq \theta_e$ $\wedge 360c \leq 720\theta P_M + 720P_{e,max} - 360\theta_f(T)^2 P_{e,max}$ $+30\theta_{f}(T)^{4}P_{e,max}-\theta_{f}(T)^{6}P_{e,max}-360\omega_{f}(T)^{2}$ $plant \equiv \theta' = \omega, \omega' = a, \sin \theta' = \omega \cos \theta, \cos \theta' = -\omega \sin \theta, t' = 1$ $H \equiv \sin^2 \theta + \cos^2 \theta = 1$ $req \equiv \theta \leq \theta_{max}$


Embedding material as identification markers

Embedded materials are unknown to third party

> Micro/nano materials are invisible to naked eyes

- Unknown materials averts the scanning attempts without correct modality
- Embedding material as "barcode" for unique identification





Previous study: identification markers in blood tests



Microbeads with different size and concentration can identify multiple test results



Embedding markers



Raman spectroscopy

- Monochromatic light source
- Measure the shifted frequency of incident light

SERS substrate

Surface-enhanced Raman spectroscopy (SERS) using gold nanorods (GNRs) or 3,3'-Diethylthiatricarbocyanine iodide (DTTCI) as enhancers

Embedded markers selection based on the availability of scanning modalities





Raman spectroscopy

- Enhanced spectral response of Silicon using GNRs and dye (DTTCI) in SERS
- Spectral responses of control ABS and embedded ABS show
 - Unadulterated ABS has more concentrated spectral response
 - Spreading response due to uncontrolled concentration of embedded markers

300um depth limitation



MicroCT

- ➤ Higher depth scan for 3D object reconstruction
- ➤ Steel saturated filament is used for higher contrast

MicroCT scan of ABS cylindrical tube with embedded GNRs



MicroCT scanning: ABS filament



MicroCT scan of ABS cylindrical tube with embedded GNRs



Embedded GNRs with high reflection

The precise placement of GNRs is out of scope for the current work

Quantitative spatial analysis also shows error



Scadman: Cyber-physical Control Flow Integrity (Submitting to USENIX Security 2018)





Hybrid Program Representation

Final Complete SMIB Hybrid Program *init* \Rightarrow [{*ctrl*; *plant*&*H*}*](*reg*) *init* $\equiv P_M = 1 \wedge P_{e,max} = \frac{3}{2} \wedge \omega = 0 \wedge \theta = \arcsin(\frac{P_M}{P_{e,max}})$ $\bigwedge \theta_{max} = \pi - \theta \bigwedge \sin \theta = \frac{P_M}{P_{e,max}} \bigwedge \cos \theta = \frac{\sqrt{P_{e,max}^2 - P_M}}{P_{e,max}}$ $\bigwedge c = 2P_M \theta_{max} - 2P_{e,max} \cos(\theta)$ $ctrl \equiv \{(a := P_M - Pe, max\sin(\theta)) \cup (a := P_M)\};$ t := 0;d0 := d;v0 := v;T := *; $?T \geq 0 \land \theta_f(T) \leq \theta_e$ $\wedge 360c \le 720\theta P_M + 720P_{e,max} - 360\theta_f(T)^2 P_{e,max}$ $+30\theta_f(\overline{T})^4 P_{e,max} - \theta_f(\overline{T})^6 P_{e,max} - 360\omega_f(\overline{T})^2$ $plant \equiv \theta' = \omega, \omega' = a, \sin \theta' = \omega \cos \theta, \cos \theta' = -\omega \sin \theta, t' = 1$ $H \equiv \sin^2 \theta + \cos^2 \theta = 1$ $req \equiv \theta \leq \theta_{max}$



HyPLC: Hybrid PLC Program Translation for Verification



Tibial Implant MicroCT Scan



Tell Me More Than Just Assembly! Reversing Cyber-physical Execution Semantics of Embedded IoT Controller Software Binaries

Pengfei Sun⁺, Luis Garcia* and Saman Zonouz⁺ ⁺Rutgers University, *University of California, Los Angeles

We live in a cyber-physical world...



* https://bricz.com/oms-integration/internet-of-thingsiot-and-cloud-computing-concept-smart-city-cyber-physical-systemscps/

...and attacks are increasingly cyber-physical







Reverse Engineering



8048094:

8048095: mov

push

ebp



Mismo: Domain-specific Reverse Engineering Framework

- Propose a domain-specific reverse engineering framework to extract high-level algorithmic control- and data-flow semantics from embedded binary executables.
- Introduce a semantic mapping using dynamic binary analysis and symbolic comparison of the mathematical and binary expressions to fill the semantic gap between high-level algorithm descriptions and low-level stripped binary segments.

Domain Knowledge (Cyber-physical System)



Current Solutions

- IDA PRO
- OllyDbg
- Ninja
- Snowman
- Reward (NDSS'10)
- TIE (NDSS'11)
- Howard (NDSS'11)
- ReViver (ACSAC'16)
- BinDiff (SSTIC'05)
- BinJuice (PPREW '13)
- Blex (Usenix'14)

Disassembler, Decompiler, Static and dynamic analysis Data structure definition, Data structure memory instance Binary similarity checking





Symbolic Controller Abstract Syntax Trees



Algorithmic Mathematical Abstract Syntax Trees

Formal Symbolic Equivalence Check



Annotated Disassembly











Symbolic expression



Symbolic Controller Abstract Syntax Trees







Annotated Disassembly

; Attributes sub_8EC4	: bp-based frame fpd=0x3C	IDA Pro disassembly result									
<pre>var_3C= -0x3 var_34= -0x3 var_2C= -0x2 var_24= -0x2 var_1C= -0x1 var_14= -0x1 PUSH VPUSH SUB ADD LDD</pre>	C 4 C 4 4 4 4 4 4 4 4 4 4 4 4 7, LR} {D8} SP, \$P, #0x34 R7, SP, #0 PA = (here 1 = 0,0) PA = (here 1 = 0,0)	Mismo's extracted semantic information									
ADD LDR LDR VLDR LDR VLDR VMUL.F64 LDR VLDR LDR VLDR LDR VLDR VLDR VLDR VLDR VLDR VADD.F64	R4, -(uword_1B000 - 0x8ED4) R4, PC ; dword_1B000 R3, =(off_1B09C - 0x1B000) R3, [R4,R3] ; unk_1B258 ; D6, [R3] ; R3, =(off_1B0B4 - 0x1B000) R3, [R4,R3] ; unk_1B5E8 ; D6, D6, D7 ; R3, =(off_1B09C - 0x1B000) R3, [R4,R3] ; unk_1B258 ; D5, [R3, #0x10] ; R3, =(off_1B0C0 - 0x1B000) R3, [R4,R3] ; unk_1B640 ; D7, [R3] ; D7, [R3] ; D7, D5, D7 ; D8, D6, D7 ;	<pre></pre>									
LDR	R3, = (off_1B09C - 0x1B000) R3 [R4 R3] : unk 1R258 :	struct pointer unk 18258									

Evaluation

Accuracy

Category	Vendor	Control Algorithm				1	#firmwares	Data Accuracy (%)	Code Accuracy (%)	
		BB	KF	PF	PID	PWM	^m m m war es	Data Accuracy (%)	Coue Accuracy (%)	
Drone	Bitcraze		1		1	1	38	100.00	96.40	
	Ardupilot						168	78.57	86.96	
	DJI						66	100.00	93.69	
	3D Robotics				-		327	78.57	86.96	•••
	Cleanflight				-		48	71.43	50.26	c variables
	Fluoreon				-		1	77.78	48.70	c variables
	Eachine				-		1	77.78	48.70	
	Paparazzi				\checkmark		53	77.78	86.14	
	Cheerson				\checkmark		169	84.29	91.56	
Automotive	Baidu		 ✓ 		-		2	100.00	93.67	
	PolySync				-		3	100.00	97.01	
	Microsoft				\checkmark		1	100.00	100.00	
	Tier IV				\checkmark		11	100.00	89.47	
	Udaticy			\checkmark	\checkmark		2	100.00	97.14	
3D Printer	LulzBot						22	90.91	92.86	
	Makerbot				-		19	88.89	63.81	lione
	Repetie				-		6	100.00	82.96	LIONS
	Printrbot				-		22	90.91	92.86	
	BCN3D						15	81.82	50.26	
	Robo3D				\checkmark		1	90.91	92.86	
	Teacup						1	100.00	93.24	
	Solidoodle						2	90.91	92.86	
Robotics -	ROS		1	1	 Image: A set of the set of the	 ✓ 	62	88.89	94.20	
	Robotiq						1	100.00	98.64	
	LinuxCNC				-		145	53.85	43.34	
	Drake				-		8	85.71	87.38	
Smart Home	SmartPID				1		2	100.00	100.00	
	Particle						87	100.00	96.81	
	MBED						147	100.00	100.00	
Linux Kernel	Linux Kernel				1		833	100.00	100.00	
Total/Average	30						2,263	89.82	84.96	
• Binary vulnerability assessment

- Binary vulnerability assessment
- Memory forensics analysis

- Binary vulnerability assessment
- Memory forensics analysis
- Sensitive code and data segment protection

- Binary vulnerability assessment
- Memory forensics analysis
- Sensitive code and data segment protection
- Correct algorithm implementation verification

- Binary vulnerability assessment
- Memory forensics analysis
- Sensitive code and data segment protection
- Correct algorithm implementation verification
- Binary-level software similarity measures

Compare with Snowman

Source Code	Snowman Reversed Result	MISMO Reversed Result
<pre>typedef struct { double windup_guard; double proportional_gain; double integral_gain; double derivative_gain; double prev_input; double int_error; double control; double prev_steering_angle; } PID;</pre>	<pre>signed int v6; // r3@2 double v19; // [sp+0h] [bp+0h]@1 double v20; // [sp+8h] [bp+8h]@8 int v21; // [sp+1Ch] [bp+1Ch]@1</pre>	<pre>struct { 0: double SymVar; 8: double Kp; 10: double Ki; 18: double Kd; 20: double prev_measured_value; 28: double integral; 30: double output; }</pre>
diff = ((input - pid-> prev_input)/dt);	-R3 = v21; asm { VLDR D7, [R3,#0x20] VLDR D6, [R7,#0x4C+var_44] VSUB.F64 D6, D6, D7 VLDR D7, [R7,#0x4C+var_4C] VDIV.F64 D7, D6, D7 VSTR D7, [R7,#0x4C+var_24] }	reg_D6 = measured_value - previous_measured_value; reg_D7 = reg_D6/dt;

Linux Kernel Bug



Attack Control Algorithm





(a) Car crash visualization using the autonomous controller. (b) Modified gain parameter of the controller causes the crash.

Conclusions & QA

- A general framework to extract semantic information of an embedded firmware binaries with respect to its associated high-level control algorithm.
- Using dynamic binary analysis and symbolic comparison of the mathematical and binary expressions to fill the semantic gap between high-level algorithm descriptions and low-level stripped binary segments.

Thank You

Questions? Email: garcialuis@ucla.edu

PID

$$u(t) = \mathrm{MV}(t) = K_\mathrm{p} e(t) + K_\mathrm{i} \int_0^t e(au) \, d au + K_\mathrm{d} \, rac{de(t)}{dt},$$

where

 $K_{
m p}$ is the proportional gain, a tuning parameter,

 $K_{\rm i}$ is the integral gain, a tuning parameter,

 $K_{\rm d}$ is the derivative gain, a tuning parameter,

 $e(t) = \mathrm{SP} - \mathrm{PV}(t)$ is the error (SP is the setpoint, and PV(*t*) is the process variable),

、 /

t is the time or instantaneous time (the present),

 τ is the variable of integration (takes on values from time 0 to the present *t*).

Kalman Filter

Predict [edit]	
Predicted (a priori) state estimate	$\hat{\mathbf{x}}_{k k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1 k-1} + \mathbf{B}_k \mathbf{u}_k$
Predicted (a priori) error covariance	$\mathbf{P}_{k k-1} = \mathbf{F}_k \mathbf{P}_{k-1 k-1} \mathbf{F}_k^T + \mathbf{Q}_k$
Update [edit]	
Innovation or measurement pre-fit residual	$ ilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k k-1}$
Innovation (or pre-fit residual) covariance	$\mathbf{S}_k = \mathbf{R}_k + \mathbf{H}_k \mathbf{P}_{k k-1} \mathbf{H}_k^T$
Optimal Kalman gain	$\mathbf{K}_k = \mathbf{P}_{k k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$
Updated (a posteriori) state estimate	$\hat{\mathbf{x}}_{k k} = \hat{\mathbf{x}}_{k k-1} + \mathbf{K}_k ilde{\mathbf{y}}_k$
Updated (<i>a posteriori</i>) estimate covariance	$\mathbf{P}_{k k} = \left(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k ight) \mathbf{P}_{k k-1} \left(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k ight)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T$
Measurement post-fit residual	$ ilde{\mathbf{y}}_{k k} = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k k}$

The formula for the updated (*a posteriori*) estimate covariance above is valid for any gain \mathbf{K}_k and is sometimes called the **Joseph form**. For the optimal Kalman gain the formula further simplifies to $\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$, in which form it is most widely used in applications. However, one must keep in mind, that it is valid only for the optimal gain that minimizes the residual error. Proof of the formulae is found in the *derivations* section.